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On Measures with Values in Partially Ordered Spaces[†]

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Let $(X, \mathcal{S})$ be a measurable space and X be of the power of the continuum. Let μ be a measure on X with values in partially ordered Abelian group G. Using g-regularity of G and under the continuum hypothesis, the analogy of the *Banach problem* for μ is solved in a case when lattice structure of G is not supposed (this is the case, e.g., in the ordered vector space of Hermitian operators in a Hilbert space).

1. INTRODUCTION

It is well known that there does not exist on the class of all subsets of a given set X a finite, nontrivial, σ-additive measure μ, i.e., $\mu:2^X \to \mathbb{R}$, that assigns 0 to each singleton of X. If X is a countable set, the statement is obvious; if X has the power of the continuum, the statement has been proved in ref. 2 under the assumption of the continuum hypothesis.

An analogy of this classical *Banach problem* for measures having their values in the Stone algebra $C(S)$ of continuous functions on extremally disconnected compact Hausdorff space S was given in ref. 5. The result was achieved under the additional assumption that each meager subset of S is nowhere dense. This topological property of S is equivalent to the algebraic property of $C(S)$, which is the weak (σ, ∞)-distributivity (see ref. 9 for details). In ref. 7 the problem was solved for measures taking their values in weakly σ-distributive vector lattices or lattice groups.

[†] This paper is dedicated to the memory of Prof. Gottfried T. Rüttimann.

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The aim of this paper is to avoid the lattice structure of the range space and suggest the property of the ordered space that can substitute for the weak σ-distributivity which was so useful in the lattice case.

2. *G*-VALUED (SUB)MEASURES

Basic notions and a notation in this paper are used in the sense of refs. 2–4. As mentioned above, the range space of the measure μ is a partially ordered Abelian group G, i.e., a commutative group $(G, +)$ partially ordered by a reflexive, antisymmetric, and transitive relation $\leq$ which is consistent with the group structure, i.e., $a \leq b$ implies $a + c \leq b + c$ for any $a, b, c \in G$. Of course, if, moreover, G is a real vector space, then $a \leq b$ implies $\lambda.a \leq \lambda.b$ for any positive scalar λ.

In the 1970s J. D. M. Wright developed the theory of vector lattice-valued measures whose σ-additivity is defined via the order structure of the range space [8–10]. In ref. 10 he studied the concept of a V-valued measure in the nonlattice case when V is a monotone σ-complete partially ordered vector space. As the vector structure itself is not essential for the concept of a measure, we introduce the G-valued measure as follows.

Definition 2.1. Let $(\Omega, \mathcal{S})$ be a measurable space, G be a monotone σ-complete, partially ordered Abelian group, and μ be a map $\mu: \mathcal{S} \to G$. μ is said to be a G-valued measure on $(\Omega, \mathcal{S})$ if:

(1) $\mu(\emptyset) = 0$ and $\mu(A) \geqslant 0$ for any $A \in \mathcal{S}$.

(2) μ is σ-additive, i.e.,

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigvee \left\{\sum_{i=1}^{n} \mu(A_i) \mid n = 1, 2, \ldots\right\}$$

whenever (A_i) is a disjoint sequence of elements in $\mathcal{S}$.

For brevity, we use $\sum_{i=1}^{\infty} a_i$ instead of $\vee\{\sum_{i=1}^{n} a_i \mid n = 1, 2, \ldots\}$. It is easy to see that a G-valued measure is σ-subadditive and continuous from below at every $A \in \mathcal{S}$.

Definition 2.2. Let $(\Omega, \mathcal{S})$ be a measurable space, G be a monotone σ-complete, partially ordered Abelian group, and μ be a map $\mu: \mathcal{S} \to G$. μ is said to be a G-valued submeasure on $(\Omega, \mathcal{S})$ if:

(1) $\mu(\emptyset) = 0$.

(2) $\mu(A) \leqslant \mu(B)$ whenever $A \subset B$, $A, B \in \mathcal{S}$.

(3) $\mu(A \cup B) \leqslant \mu(A) + \mu(B)$ for any $A, B \in \mathcal{S}$.

(4) $\wedge \mu(A_n) = 0$ whenever (A_n) is a monotone decreasing sequence in $\mathcal{S}$ for which $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

Lemma 2.3. Let μ be a G-valued submeasure on $(\Omega, \mathcal{S})$. Then:

(1) μ is continuous from below at any $A \in \mathcal{S}$.
(2) μ is σ-subadditive.

3. REGULARITIES OF ORDERED SPACES

As mentioned in the Introduction, in the lattice case the key role in the solution of the Banach problem is played by the condition of the weak σ-distributivity of the range space of μ. We recall that a vector lattice V is said to be weakly σ-distributive if whenever (a_{ij}) is bounded from above, such that $a_{ij} \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$, then

$$\bigwedge \left\{ \bigvee_{i=1}^{\infty} a_{i\varphi(i)} \middle| \varphi: \mathbb{N} \to \mathbb{N} \right\} = 0$$

It is worth mentioning that this concept has a crucial role in the measure extension property of vector lattices [8, Theorem T]. For the case when G is a monotone σ-complete Abelian group we substitute the above concept for the concept of the g-regularity of G.

A monotone σ-complete, partially ordered Abelian group G is said to be a g-regular group if

$$\bigwedge \left\{ \sum_{i=1}^{\infty} a_{i\varphi(i)} \middle| \varphi: \mathbb{N} \to \mathbb{N} \right\} = 0$$

whenever (a_{ij}) is a bounded, double sequence in G such that $a_{ij} \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$. For brevity, notation $\Sigma_{i=1}^{\infty} a_{i\varphi(i)}$ stands for $\bigvee \{\Sigma_{i=1}^{n} a_{i\varphi(i)} | n = 1, 2, \ldots\}$, so that $\Sigma_{i=1}^{\infty} a_{i\varphi(i)} = \infty$ when $\{\Sigma_{i=1}^{n} a_{i\varphi(i)}\}$ is not bounded from above. We point out that the property of G to be g-regular implicitly requires that if (a_{ij}) is bounded and $a_{ij} \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$, then there exists $\varphi: \mathbb{N} \to \mathbb{N}$ such that $\{\Sigma_{i=1}^{n} a_{i\varphi(i)}\}$ is bounded from above so that $\Sigma_{i=1}^{\infty} a_{i\varphi(i)}$ is in G. The next example exhibits a vector lattice in which this is not always the case.

Example 3.1. Let l_∞ be the sequence space whose elements are all bounded real sequences. Let the ordering be coordinatewise. Consider a double sequence (a_{ij}) defined by

$$a_{ij}(k) = 0 \quad \text{for} \quad k = 1, 2, \ldots, j-1 \quad \text{and each} \quad i, j \in \mathbb{N}$$

$$a_{ij}(k) = 1 \quad \text{for} \quad k = j, j+1, \ldots \quad \text{and each} \quad i, j \in \mathbb{N}$$

It can be easily verified that for every $\varphi: \mathbb{N} \to \mathbb{N}$ the sequence $(\Sigma_{i=1}^{n} a_{i\varphi(i)})$ is not bounded from above.

It is easy to see that in the class of lattice groups the condition of g-regularity is strictly stronger than weak σ-distributivity (see ref. 6 for the next relationships). On the other hand, the next proposition gives new light on the concept of g-regularity of the lattice groups.

Proposition 3.2. Let G be a σ-complete lattice group. Let (a_{ij}) be a bounded, double sequence in G, $a_{ij} \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$. If there exists $\varphi^0: \mathbb{N} \to \mathbb{N}$ such that the sequence $\{\sum_{i=1}^{n} a_{i\varphi^0(i)}\}$ is bounded, then

$$\bigwedge\left\{\sum_{i=1}^{\infty} a_{i\varphi(i)} \middle| \varphi: \mathbb{N} \to \mathbb{N}\right\} = 0$$

Proof. Let b be a lower bound for $\{\sum_{i=1}^{\infty} a_{i\varphi(i)} | \varphi: \mathbb{N} \to \mathbb{N}\}$. Set $c = b \vee 0$. Obviously, c is also lower bound of $\{\sum_{i=1}^{\infty} a_{i\varphi(i)} | \varphi: \mathbb{N} \to \mathbb{N}\}$ and $c \geqslant 0$. However, we show that $c > 0$ leads to a contradiction, so that $c = 0$ and it gives $b \leqslant 0$. This proves that 0 is the greatest lower bound of $\{\sum_{i=1}^{\infty} a_{i\varphi(i)} | \varphi: \mathbb{N} \to \mathbb{N}\}$.

Now assuming that positive c is a lower bound of $\{\sum_{i=1}^{\infty} a_{i\varphi(i)} | \varphi: \mathbb{N} \to \mathbb{N}\}$, we wish to derive a contradiction. According to Theorem 4 in ref. 1, for G there exist a Stone space E (i.e., compact Hausdorff and extremally disconnected) and a lattice group isomorphism h of G onto a sublattice group of the vector lattice $\mathcal{F}(E)$ of all almost finite continuous functions on E which preserves all suprema and infima in G, i.e., if $x_0 = \wedge x_\alpha$ in G, then $h(x_0) = \wedge h(x_\alpha)$ in $\mathcal{F}(E)$.

Applying this result to (a_{ij}), we get $h(a_{ij}) \searrow 0$ $(j \to \infty)$, for each $i \in \mathbb{N}$ in $\mathcal{F}(E)$. According to Lemma 2.2 in ref. 9, the set $A_i = \{x \in E$: in $f\{h(a_{ij})(x): j = 1, 2, \ldots\} > 0\}$ is a meager set (i.e., the set of the first category). Of course, $\cup_{i=1}^{\infty} A_i$ is meager and for every $x \in E \setminus \cup_{i=1}^{\infty} A_i$ we have $h(a_{ij})(x) \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$. Because c is positive, $h(c) > 0$ and there exist clopen U, $U \subset E$, and real ε, $\varepsilon > 0$, such that $h(c)(x) > \varepsilon$ for all $x \in U$. According to the Baire category theorem, $U \setminus \cup_{i=1}^{\infty} A_i$ is nonempty, so that there is $x_0 \in U \setminus \cup_{i=1}^{\infty} A_i$.

Since $h(a_{ij})(x_0) \searrow 0$ $(j \to \infty)$ for each $i \in \mathbb{N}$, it is possible to define $\psi: \mathbb{N} \to \mathbb{N}$ such that $\varphi^0(i) \leqslant \psi(i)$ for every $i \in \mathbb{N}$ so that $\{\sum_{i=1}^{n} a_{i\psi(i)}\}$ is bounded in G (i.e., $\sum_{i=1}^{\infty} a_{i\psi(i)} \in G$) and

$$\sum_{i=1}^{\infty} h(a_{i\psi(i)})(x_0) \leq \varepsilon$$

Let us summarize; c is a lower bound of $\{\sum_{i=1}^{\infty} a_{i\varphi(i)} | \varphi: \mathbb{N} \to \mathbb{N}\}$, so that $h(c)$ is a lower bound of $\{\sum_{i=1}^{\infty} h(a_{i\varphi(i)}) | \varphi: \mathbb{N} \to \mathbb{N}\}$. But $h(c)(x_0) > \varepsilon$, whereas $\sum_{i=1}^{\infty} h(a_{i\psi(i)})(x_0) \leq \varepsilon$. Because the ordering in $\mathcal{F}(E)$ is pointwise, we get the desired contradiction.

It is shown in ref. 6 that the g-regularity in the class of Riesz spaces is strictly less restrictive than the condition regularity in the sense of Kantorovich (i.e., less restrictive than so called *strong Egoroff* property of Riesz space; for details, see ref. 4, Chapter 10).

4. MAIN RESULT

Theorem 4.1. Let us assume the continuum hypothesis. Let $(\Omega, \mathcal{S})$ be a measurable space and let E be a set of the power of the continuum. Let G be a monotone σ-complete, g-regular, partially ordered Abelian group and μ a G-valued measure on $\mathcal{S}$. If $\{A_x : x \in E\}$ is a family of pairwise disjoint sets in $\mathcal{S}$ such that $\cup\{A_x\colon x \in F\} \in \mathcal{S}$ for any $F \subset E$, then

$$\mu(\cup\{A_x\colon x \in E\}) = \vee\{\mu(\cup\{A_x\colon x \in I\}) | I \subset E, I \text{ is finite}\}$$

Proof. Let us set $b = \mu(\cup\{A_x\colon x \in E\})$ and denote by $\mathcal{I}$ the system of all finite subsets of E, i.e., $\mathcal{I} = \{I | I \subset E, I \text{ is finite}\}$. It is evident that $\mu(\cup\{A_x\colon x \in I\}) \leqslant b$ for any $I \in \mathcal{I}$. We are to prove that if $c \in G$ and such that $\mu(\cup\{A_x\colon x \in I\}) \leqslant c$ for any $I \in \mathcal{I}$, then $b \leqslant c$ (i.e., b is the least upper bound of $\{\mu(\cup\{A_x\colon x \in I\}) | I \in \mathcal{I}\}$.

Now we use the *Banach–Kuratowski* theorem (e.g., ref. 4, paragraph 75), which states that under the continuum hypothesis there exists a double sequence (E_{ij}) of subsets of E such that:

(1) $E_{ij} \subset E_{ij+1}$ for each $i, j \in \mathbb{N}$.
(2) $\cup_{j=1}^{\infty} E_{ij} = E$ for each $i \in \mathbb{N}$.
(3) For any $\varphi\colon \mathbb{N} \to \mathbb{N}$ the intersection $\cap_{i=1}^{\infty} E_{i\varphi(i)}$ is a countable set.

Let us set $B_{ij} = \cup\{A_x\colon x \in E \setminus E_{ij}\}$ and $a_{ij} = \mu(B_{ij})$. From (1) we get that $B_{ij} \supset B_{ij+1}$ for each $i, j \in \mathbb{N}$. Due to (2) we have $B_{ij} \searrow \emptyset$ and the continuity of μ gives $a_{ij} \searrow 0$ ($j \to \infty$) for each $i \in \mathbb{N}$. According to (3), for any $\varphi\colon \mathbb{N} \to \mathbb{N}$ it is possible to enumerate points of $\cap_{i=1}^{\infty} E_{i\varphi(i)}$ so $\cap_{i=1}^{\infty} E_{i\varphi(i)} = \{x_1, x_2, x_3, \ldots\}$.

From the fact that $c \in G$ is the upper bound of $\{\mu(\cup\{A_x\colon x \in I\}) | I \in \mathcal{I}\}$ we get $\mu(A_{x_1} \cup A_{x_2} \cup \ldots \cup A_{x_n}) \leqslant c$ for every $n \in \mathbb{N}$ and from the continuity of μ we have $\mu(A_{x_1} \cup A_{x_2} \cup \ldots \cup A_{x_n} \ldots) \leqslant c$. From now on

$$\begin{aligned} b - c &\leq \mu(\cup\{A_x\colon x \in E\}) - \mu(\cup\{A_{x_i}\colon i = 1,2,\ldots\}) \\ &= \mu(\cup\{A_x\colon x \in E\}) - \mu(\cup\{A_x\colon x \in \cap_{i=1}^{\infty} E_{i\varphi(i)}\}) \\ &= \mu(\cup\{A_x\colon x \in E \setminus \cap_{i=1}^{\infty} E_{i\varphi(i)}\}) \\ &= \mu(\cup\{A_x\colon x \in \cup_{i=1}^{\infty} (E \setminus E_{i\varphi(i)}\}) \end{aligned}$$

$$\leq \sum_{i=1}^{\infty} \mu(\cup\{A_x: x \in E\setminus E_{i\varphi(i)}\}) = \sum_{i=1}^{\infty} \mu(B_{i\varphi(i)})$$

We derive that $b - c \leq \sum_{i=1}^{\infty} a_{i\varphi(i)}$ for any $\varphi: \mathbb{N} \to \mathbb{N}$. From the g-regularity of G we get $b - c \leq 0$, i.e., $b \leq c$, and this completes the proof.

In the chain of (in)equalities we used the fact that if $F \subset E$, then $\mu(\cup\{A_x: x \in E\}) - \mu(\cup\{A_x: x \in F\} = \mu(\cup\{A_x: x \in E\setminus F\}$, which holds due to the fact that $A_x \cap A_y = \emptyset$ for $x \neq y$. If μ is a submeasure, we have the inequality in the above equality. With respect to Lemma 2.3, Theorem 4.1 holds also in the case when μ is a submeasure.

Theorem 4.2. Let us assume the continuum hypothesis. Let $(X, \mathcal{S})$ be a measurable space and let X be a set of the power of the continuum. Let G be a monotone σ-complete, g-regular, partially ordered Abelian group and μ be a G-valued submeasure on $\mathcal{S}$ such that $\mu(\{x\}) = 0$ for all $x \in X$. If there exists a set $E \in \mathcal{S}$ such that $\mu(E) > 0$, then there exists $F \subset X$ such that $F \notin \mathcal{S}$.

Proof. If $E \in \mathcal{S}$ for every $E \subset X$, then $E = \cup\{\{x\}: x \in E\}$, and according to Theorem 4.1, we get

$$\begin{aligned} \mu(E) &= \vee\{\mu(\cup\{\ \{x\}: x \in I\})\big| I \subset E, I \text{ is finite } \} \\ &= \vee\{0 \big| I \subset E, I \text{ is finite } \} = 0 \end{aligned}$$

so that μ is trivial—a contradiction.

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